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BASIC PROBLEMS OF A-POSTERIORI ERROR ESTIMATION

by

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Technical Note BN-1131

February 1992

INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

92-09423

92

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Note BN-1131	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) BASIC PROBLEMS OF A-POSTERIORI ERROR ESTIMATION		5. TYPE OF REPORT & PERIOD COVERED Final Life of Contract
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) I. Babuska[1] - L. Plank[2] - R. Rodriguez[3]		8. CONTRACT OR GRANT NUMBER(s) [1]ONR N00014-90-J-1090 [2]CCR-88-20279 (NSF) [3]CCR-88-20279 (NSF)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Physical Science and Technology University of Maryland College Park, MD 20742-2431		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE February 1992 13. NUMBER OF PAGES 28
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (

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S/N 0102-LF-014-6601

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BASIC PROBLEMS OF A-POSTERIORI ERROR ESTIMATION

I. Babuška,[1] L. Plank,[2] R. Rodriguez[3]

Abstract. This paper addresses basic questions regarding reliability of a-posteriori error estimation. It analyzes in detail an a-posteriori estimator for linear elements on triangular meshes. It gives the bounds of the effectivity index depending on the geometry of the triangles and smoothness of the approximated solution. The theoretical results are in concordance with the results of our numerical experiments. The second part analyzes in one dimensional setting a one parametric family of the estimators and their optimal selection for adaptively constructed meshes.

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1. INTRODUCTION

During the last ten years, starting essentially with [2-5], great progress has been made in the theory and practice of a-posteriori error estimation. A-posteriori error estimation is then directly related to adaptive construction of the meshes. For survey of various results, we refer to [6], [7], [15], [24] and the references therein. Many estimators were developed and analyzed either by numerical experimentation, mathematical theory, or both. Nevertheless, there is no sufficiently detailed mathematical understanding of various estimators, especially regarding their reliability (e.g., bounds of the effectivity indices, etc.).

In this paper, we analyze in detail a particular a-posteriori error estimator for linear triangular elements. We show the dependence of its effectivity index on various factors such as geometry and smoothness of the solution. We present theoretical results and numerical benchmark computations. Further, we analyze a one parametric family of a-posteriori error estimators in one dimensional setting. We show that for adaptively constructed meshes (which are equilibrated), asymptotic exactness of the estimator can be achieved for unsmooth solutions by an optimal choice of the parameter.

2. BASIC NOTIONS IN THE THEORY OF THE A-POSTERIORI ERROR ESTIMATION FOR A MODEL PROBLEM

Let $\Omega \subset \mathbb{R}^2$ be a polygon with the boundary Γ. Denote by $H = H_0^1(\Omega)$ the standard Sobolev space, and consider the bilinear form $B(u,v)$ on $H \times H$:

$$B(u,v) = \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dxdy,$$

with

$$\|u\|_E = (B(u,v))^{1/2}.$$

Let $f \in L_2(\Omega)$ and

$$F(v) = \int_\Omega fv\,dxdy \in H'$$

be a functional on H. Then, by $u_0 \in H$ we will denote the (exact) solution of the problem $\mathcal{P}$

$$B(u_0,v) = F(v),\ \forall\ v \in H. \tag{2.1}$$

Obviously, u_0 is uniquely defined by (2.1)

Let $\mathcal{M}$ be a class of triangularizations of Ω; by $\mathcal{T} \in \mathcal{M}$ we denote the triangular mesh and by $\tau = \mathcal{T}$ the triangular element. Let $S(\mathcal{T}) \subset H$ be the set of all functions which are linear on every element $\tau \in \mathcal{T}$. Then, as usual, by $u_S \in S(\mathcal{T})$ we denote the finite element solution of the problem $\mathcal{P}$:

$$B(u_S,v) = B(u_0,v) = F(v), \quad \forall\ v \in S(\mathcal{T}). \tag{2.1}$$

We associate to every $\tau \in \mathcal{T}$ an <u>error</u> <u>indicator</u> $\eta(\tau) \geq 0$, which value depends only on u_S and f on τ and on the neighboring elements. The concrete form of some error indicators will be given below.

We define the <u>error</u> <u>estimator</u>

$$\mathcal{E}(\mathcal{T}) = \left(\sum_{\tau\in\mathcal{T}} \eta^2(\tau)\right)^{1/2}. \tag{2.2}$$

We will be interested in estimating the error in the energy norm, i.e. we assume that $\mathcal{E}(\mathcal{T}) \cong \|u_0-u_S\|_E$. Further we define the effectivity index $\Theta(\mathcal{T})$ of the estimator $\mathcal{E}$:

$$\Theta(\mathcal{T}) = \frac{\mathcal{E}(\mathcal{T})}{\|u_0-u_S\|_E}. \tag{2.3}$$

The error indicator, estimator and effectivity index depend obviously on u_0, f and $\mathcal{T}$.

Let $\mathcal{S} \subset H$ be a set of the (exact) solutions under consideration. We will call the estimator $\mathcal{E}(\mathcal{T})$ a $(\mathcal{M},\mathcal{S},\kappa_1,\kappa_2)$-<u>proper</u> if for any $u_0 \in \mathcal{S}$ we have

(2.4) $$\kappa_1(\mathcal{M},\mathcal{S}) \le \Theta \le \kappa_2(\mathcal{M},\mathcal{S}).$$

The estimator will be called <u>asymptotically</u> $(\mathcal{M},\mathcal{S})$-<u>exact</u> if for any $u_0 \in \mathcal{S}$ and any sequence $\mathcal{T}_i \in \mathcal{M}$, $i = 1,2\ldots$ such that

(2.5) $$\|u_{S(\mathcal{T}_i)} - u_0\|_E \to 0$$

as $i \to \infty$ we have

(2.6) $$\Theta(\mathcal{T}_i) \to 1.$$

If $\Theta \ge 1$, i.e. $\kappa_1 \ge 1$, then $\mathcal{E}$ will be called $(\mathcal{M},\mathcal{S})$-<u>upper</u> estimator and if $\Theta \le 1$, i.e. $\kappa_2 \le 1$, then $\mathcal{E}$ will be called $(\mathcal{M},\mathcal{S})$-lower estimator. We note that κ_1, κ_2 depend on $(\mathcal{M},\mathcal{S})$. We can also consider the case that $\mathcal{S}$ consists of one element only, namely u_0.

If κ_1,κ_2 would be known, then an upper , respectively, lower estimator can be obtained by multiplication of $\mathcal{E}$ by $\frac{1}{\kappa_1}$, respectively, $\frac{1}{\kappa_2}$.

An asymptotically exact error estimator can be easily constructed by exploiting the superconvergence effect of the derivatives of the finite element solution. A systematic study of the superconvergence phenomena in the finite elements seems to have begun in early 1970. By now many different superconvergence effects have been analyzed. For the reviews and references, we refer to [20], [27] and [30]. Most results on superconvergence of the derivatives for linear triangles are based on [22] and [23]. These ideas were used very effectively in [19]. See also [16] and [17]. Superconvergence results hold only for a very small class $(\mathcal{M},\mathcal{S})$, when the meshes are essentially uniform and the solutions are smooth. In [1] it is shown that for quadratic isoparametric elements, the results are of a very similar nature. In [18] the meshes T_h with triangles that are obtained from triangles by uniformly subdividing each element of the mesh $\mathcal{T}_{2h}$ into 4 congruent

subtriangles are analyzed. It is shown that under certain restriction on $\mathcal{T}_h$ a quadratic interpolation of the solution obtained by linear elements yields the same (asymptotically) accuracy in gradients as of the quadratic elements. Various asymptotically exact estimates were derived differently although the superconvergence is essentially present. See, e.g., [13] and references there. The superconvergence effects need a smooth solution and regular mesh. This assumption can be weakened when only superconvergence on an interior domain is analyzed [26]. See also [21] for relevant results. The interior estimates effect can be used for proving asymptotic exactness of the error estimators on patchwise uniform meshes as in [5] and [8].

Any asymptotically exact estimator is closely related to the superconvergence and hence a very stringent assumption on the class $(\mathcal{M},\mathcal{S})$ is needed. Nevertheless, it is possible to expect that the effectivity index will be a reasonable one for the class $(\mathcal{M},\mathcal{S})$ which is not too far from the one that leads to the asymptotic exactness of the estimator.

In contrast to the asymptotic exactness, the properness holds for much larger classes of $(\mathcal{M},\mathcal{S})$. Essentially it could hold for any $u_0 \in H$ defined by (2.1) and all triangulation which satisfies the minimal angle condition. Existence of κ_1, κ_2 can be proven in most cases, for example, using the ideas in [2], [5] and [8]. In Section 3 we will give the concrete values of κ_1, κ_2 for an estimator.

The aim to establish the asymptotic exactness of the estimator is often practically not important, because in most applications the solution is insufficiently smooth and the mesh used in practice is not sufficiently regular. On the other hand, the mesh is often constructed adaptively and then it is typical that the error indicators are nearly equilibrated , i.e., that we can consider subclass $\mathcal{M}^*$, such that

$$(2.7) \qquad \mathcal{M}^* = \left\{ \mathcal{T} \in \mathcal{M} \;\middle|\; \left(\max_{\tau \in \mathcal{T}} \eta(\mathcal{T}) \,/\, \min_{\tau \in \mathcal{T}} \eta(\mathcal{T}) \right) < k,\ u_0 \in \mathcal{S} \right\} .$$

In addition, $\mathcal{S}$ could be restricted to the functions which are the solution of the problem $\mathcal{P}$ with smooth f, for example, $f \in H^1(\Omega)$ and hence, u_0 has special singularity in the corners of the polygon Ω. Obviously, it is practically important to analyze the performance of an estimator for the class $(\mathcal{M}^*, \mathcal{S})$. In Section 5 we will address this question in the one dimensional setting.

Although we mentioned only a simple model problem, all the notions obviously make sense in more complicated settings, say, for example, the elasticity problem.

3. ANALYSIS OF AN ESTIMATOR

For any triangular element $\tau \in \mathcal{T}$, let $\mathcal{L}(\tau)$ be the set of its three edges. Further, let

$$\Pi_\tau f = \frac{1}{|\tau|} \int_\tau f \, dxdy$$

where by $|\tau|$ we denoted the area of τ. For each edge ℓ of the triangulation $\mathcal{T}$ set

$$\mathcal{J}_\ell = \begin{cases} \left\| \dfrac{\partial u_s}{\partial n} \right\|_\ell & \text{for } \ell \subset \Gamma \\ 0 & \text{for } \ell \subset \Gamma \end{cases}$$

$$\left\| \frac{\partial u_s}{\partial n} \right\|_\ell = \nabla\left(u_s \,/_{\tau_{OUT}}\right) \cdot n - \nabla\left(u_s \,/_{\tau_{IN}}\right) \cdot n,$$

where τ_{IN} and τ_{OUT} are the two triangles sharing the edge ℓ denoted by τ_{IN} and τ_{OUT}; n is the normal to ℓ outwards to τ_{IN}. $\left\| \frac{\partial u_s}{\partial n} \right\|$ denotes the jump of $\frac{\partial u_s}{\partial n}$ across the edge ℓ; this value is independent of the choice of

the direction of n

We define the error indicator $\eta(\tau)$:

$$\eta(\tau) = \left[|\tau|^2(\Pi_\tau f)^2 + \frac{1}{2} \int_{\ell\in\mathcal{L}(\tau)} |\ell|^2 \mathcal{J}_\ell^2\right]^{1/2} \tag{3.1}$$

where by $|\ell|$ we denoted the length of the edge ℓ. Then we have

Theorem 3.1. (See [12]).

$$C'_\alpha \mathcal{E} - C\left[\sum_{\tau\in\mathcal{T}} h^4(\tau)|f|^2_{1,\tau}\right]^{1/2} \le \|u_0 - u_\tau\|_E \le \tag{3.2a}$$

$$C^p_\alpha\, \mathcal{E} + \|u_0 - u_p\|_E + C\left[\sum_{\tau\in\mathcal{T}} h^4(\tau)|f|^2_{1,\tau}\right]^{1/2} \tag{3.2b}$$

□

In (3.1) by $|f|_{1,\tau}$, we denoted the $H^1(\tau)$ seminorm, $p \ge 2$ integer and $\|u_0 - u_p\|_E$ is the error of the finite element solution using elements of degree p. Constants C'_α and C^p_α are constants depending on the minimal angle α of the triangulation.

Remark. The expression (3.2) could be written in more general form and also for general boundary conditions, but we will not do it here (for more, see [12]). The ideas of the proof of (3.2b) are also partially related to [14] and [25].

Constants C'_α and C^p_α can be computed (see [12]) and we get

$$0.548(\log p)^{1/2} \sin^{-1/2}\left(\frac{\alpha}{2}\right) \le C^p_\alpha \le 0.813(\log p)^{1/2} \sin^{-1/2}\left(\frac{\alpha}{2}\right) \tag{3.3}$$

$$0.171 \sin^{1/2}\left(\frac{\alpha}{2}\right) \le C'_\alpha \le 0.290 \sin^{1/2}\left(\frac{\alpha}{2}\right). \tag{3.4}$$

In Table 3.1 we give values of C'_α and C^p_α for some α and p.

TABLE 3.1. Values of C'_α and C^p_α .

α°	C'_α	C^p_α			
		p = 2	p = 4	p = 6	p = 8
7.5	0.051	2.390	3.306	3.660	3.988
15.0	0.072	1.682	2.341	2.609	2.839
22.0	0.087	1.363	1.918	2.156	2.343
30.0	0.099	1.169	1.670	1.895	2.058
37.5	0.108	1.035	1.508	1.727	1.876
45.0	0.115	0.939	1.400	1.615	1.757
52.5	0.119	0.875	1.334	1.547	1.684
60.0	0.121	0.850	1.309	1.522	1.657

Let us now analyze the bounds (3.2) - (3.4). The bound is partially asymptotic because of the term involving function f without exact specification of the constant C. Nevertheless, this term is known a priori and is much smaller than $\mathcal{E}$ with the exception of the functions f, which are unlike in applications. If f is constant on the triangles, which is not uncommon in applications, then this term is not present. More important is the term $\|u_0 - u_p\|_E$, which expresses the influence of the smootheness of u_0. This term can be made arbitrarily small by selecting sufficiently large p. Nevertheless, this influences the constant C^p_α. From (3.3) and Table 3.1, we see that C^p_α is increasing with p. Hence, when the solution u_0 has the typical singularity in the neighborhood of the corner of Ω, the estimator will underestimate the true error, especially when the mesh is more or less uniform. If the mesh is refined in the area of the singularity of u_0, then lower values of p can be used and hence the estimator will be of higher quality. For more about the properties of the

p-version which are essential here, we refer to [9] and references therein.

We note that C'_α is independent of p, which indicates that the overestimation of the error by $\mathcal{E}$ is not influenced by the smootheness of the solution.

The dependence of the performance of the error estimator on the smoothness of the solution, as discussed above, is typical for any estimators used in practice. We remark that it is possible to show by the arguments in [2], [5] and [9] that C^p_α has a bound independent of p. Nevertheless, a realistic value for this bound is not available.

The expressions (3.3) and (3.4) indicate that the estimator deteriorates with the angle α. We emphasize that the estimator does not take into consideration any relation between the shape of neighboring elements. Further, the estimate uses the "worst" element although possibly very few elements of this type are present in the triangularization.

Let us remark that for the elasticity problem the constant C'_α is almost proportional to $\sin^{1/2}\left(\frac{\alpha}{2}\right)$ as for Laplace equation but $C^p_\alpha \approx \sin^{-3/2}\left(\frac{\alpha}{2}\right)$ while for the Laplace equation $C^p_\alpha \approx \sin^{-1/2}\left(\frac{\alpha}{2}\right)$. This difference is strongly related to the Korn's inequality. For more, see [12].

We mention that the estimate (3.2) is the worst case which is directly related to the $(\mathcal{M},\mathcal{S})$ properness. The question arises whether the estimate (3.2) - (3.4) is optimal, i.e., whether it is achievable for some $u_0 \in \mathcal{S}$ and $\mathcal{T} \in \mathcal{M}$. It is possible to show (see [12]) that for a special choice of the (uniform) mesh of triangles with angles $\frac{\pi}{2},\beta,\alpha = \frac{\pi}{2} - \beta$ and a particular solution u_0 we get for all $0 \le \beta \le \frac{\pi}{2}$

$$(3.5) \qquad 0.24(\text{tg } \beta)^{1/2}\,\mathcal{E} = \|u_0 - u_S\|_E.$$

Hence, if $0 < \beta \le \frac{\pi}{4}$, β is the minimal angle and we get from (3.5)

(3.6) $$K_1(\beta)\left(\sin\frac{\beta}{2}\right)^{1/2}\mathcal{E} = \|u_0 - u_S\|$$

where

$$0 < k_0 \le K_1(\beta) \le k_1, \qquad 0 \le \beta \le \frac{\pi}{4}.$$

Hence, with respect to the angle α of the triangle, the constant C'_α is optimal. For $\frac{\pi}{4} \le \beta < \frac{\pi}{2}$ the minimal angles is $\frac{\pi}{2} - \beta = \alpha$ and we get

(3.7) $$0.24\ (\mathrm{ctg}\ \alpha)^{1/2}\ \mathcal{E} = \|u_0 - u_S\|_E$$

or

$$K_2(\alpha)\left(\sin\frac{\alpha}{2}\right)^{-1/2}\mathcal{E} = \|u_0 - u_S\|_E$$

with $0 < k_0 \le K_2(\alpha) \le k_2 k_1$, $0 \le \alpha \le \frac{\pi}{4}$, and hence, with respect to the angle α also the constant C'_α is an optimal one. Obviously, it would be desirable to obtain the values of C'_α and C^p_α, which are sharp with respect to a particular class $(\mathcal{M},\mathcal{S})$.

Coming back to the effectivity index, we obviously have

$$\kappa_1 \cong \frac{1}{C^p_\alpha}, \qquad \kappa_2 \cong \frac{1}{C^1_\alpha}.$$

We note that the notion of angle α is related to the differential equation under consideration. Here we have considered only the case of problem $\mathcal{P}$ given in Section 1.

In applications we can use the error estimator $\mathcal{E}_K$ based on the error indicator $\eta_K(\mathcal{T})$ with

$$\eta_K(\tau) = K\ \eta(\tau)$$

where the constant K can be selected arbitrarily. For example, the constant K can be selected as the geometric average of C'_α and C^p_α or it can be chosen so that an upper estimate will be obtained . Another possibility is to

select K so that the error estimator will be asymptotically exact if $\mathcal{T}$ is the mesh which satisfies a proper assumption; in our case it is the mesh of equilateral triangles and $K = 0.26864$. This value will be used in the next section and the estimator will be denoted by $\mathcal{E}^*$.

We note that a correction factor K, which is sometimes used in practice, cannot influence the dispersion of the effectivity index for class solutions and meshes.

4. SOME NUMERICAL EXPERIMENTATION WITH THE ERROR ESTIMATOR (3.1)

In this section we report some computational results with the estimator $\mathcal{E}^*$ given in the previous section. For detailed results we refer to [11]. The aim of these experiments is to see how accurately the theoretical results presented in Section 3 characterize the performance of the estimator in various concrete cases.

Example 1. Consider $\Omega = (0,1) \times (0,1)$ and the problem

$$\begin{aligned} -\Delta u &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

and f is chosen such that $u_0 = \sin \pi x \sin \pi y$. Further, consider the regular 3 directional meshes that are obtained by uniform refinement of a basic mesh with the ratio $\rho = \frac{1}{n}$, $n = 1,2,4,8$ as shown in Fig. 4.1.

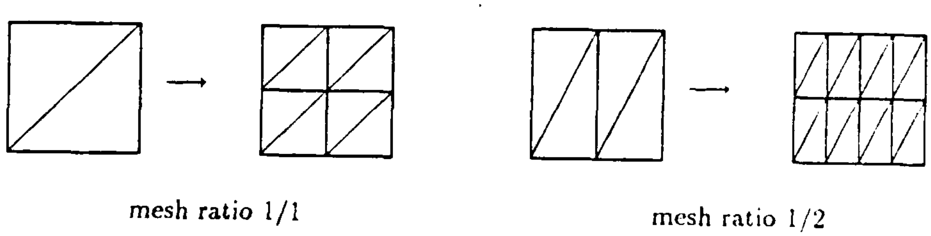

Fig. 4.1. Regular three directional mesh with the ratio $\rho = 1,2$.

TABLE 4.1. Effectivity index of the estimator for Example 1.

NUMBER OF ELEMENTS	MESH OF THE RATIO ρ			
	1/1	1.2	1/4	1/8
8	0.852			
16		0.993		
32	1.070		1.289	
64		1.183		1.804
128	1.147		1.503	
256		1.252		2.068
512	1.180		1.571	
1024		1.277		2.145
2048	1.192		1.594	
4096		1.287		2.169
UPPER BOUND	2.34	2.87	3.98	5.38
LOWER BOUND	0.29	0.21	0.15	0.12

In Table 4.1 we report the values of the effectivity index and its bounds computed from Table 3.1, for $p = 2$. From Table 4.1 we see that as $n \to \infty$, the effectivity index converges to a limiting value > 1. In this particular case, and asymptotic expansion of Θ exists. The dependence on ρ is not the same as for the bound,but relatively close. We see that the performance of the estimator deteriorates with ρ as expected from the theory. We note that $\Theta > 1$ in all cases, which,of course, does not follow from the general theory.

<u>Example 2</u>. Let $\Omega = (0,1) \times (0,1)$ as in Example 1, and let us consider the problem

$$\begin{aligned} -\Delta u &= f && \text{on } \Omega \\ u &= 0 && \text{for } x = 0,1, \quad 0 \le y \le 1 \\ \frac{\partial u}{\partial y} &= 0 && \text{for } y = 0,1, \quad 0 \le x \le 1 \end{aligned}$$

and f is such that $u_0 = \sin \pi x$. The meshes are as in Example 1, but for $\rho \le 1$ and $\rho > 1$.

This problem is not exactly the one we considered (for simplicity) in Section 3. (Nevertheless, Theorem 3.1 with the same constants hold also (see [11] and [12]).)

Table 4.2 shows the effectivity index of the error estimator.

TABLE 4.2. Effectivity index of the estimator for Example 2.

NUMBERS OF ELEMENTS		MESH OF THE RATIO ρ			
	1/1	1/2	1/4	2/1	4/1
8	1.013				
16		1.564		0.717	
32	1.107		2.258		0.507
64		1.598		0.783	
128	1.131		2.273		0.553
256		1.608		0.800	
512	1.137		2.278		0.566
1024		1.611		0.804	
2048	1.139		2.279		0.569
4096		1.612		0.806	
UPPER BOUND	2.34	2.87	3.87	2.87	3.87
LOWER BOUND	0.29	0.21	0.15	0.21	0.15

Wee see that the effectivity index converges as $n \to \infty$, as in Example 1. Also, here the asymptotic expansion of Θ exists. The observed dependence on ρ is once more relatively close to the theoretical one. We see that in this case the effectivity index can be both larger and smaller than 1 depending whether $\rho < 1$ or $\rho > 1$.

<u>Example 3</u>. Let us consider $\Omega = (-1,1) \times (-1,1)$ partitioned into 4 square subdomains ω_i, $i = 1,2,3,4$ as shown in Fig. 4.1

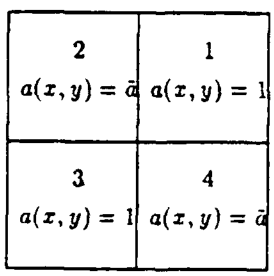

Fig. 4.2. Coefficient a for the problem of Example 3.

Let us consider the problem

$$-\nabla(a(x,y)\nabla u) = 0 \quad \text{on} \quad \Omega$$
$$a(x,y)\frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega$$

where $a(x,y)$ is constant in each ω_i, $i = 1,2,3,4$, $a(x,y) = 1$ on ω_1, ω_3 and $a(x,y) = \bar{a}$ on ω_2, ω_4. The value $\bar{a}$ is such that the exact solution u_0 in ω is

$$u_0(x,y) = r^{\alpha}(C_i \cos(\alpha\varphi) + S_i \sin(\alpha\varphi)).$$

where (κ,φ) are polar coordinates. For $\alpha = 0.5$ and $\alpha = 0.1$, we get $\bar{a} = 3 + \sqrt{8} \approx 5.828$ and $\bar{a} = 166.477...$, respectively. Although the problem addressed in this example is not exactly of the type discussed in Sections 1 and 2, the theory covers it too (see [11]).

Let us consider the uniform 3 directional meshes with the ratio $\rho = 1$ shown in Fig. 4.1. In Table 4.3 we give the effectivity index Θ.

TABLE 4.3. The effectivity index for Example 3 and uniform mesh.

NUMBER OF ELEMENTS	$\alpha = 0.5$	$\alpha = 0.1$
32	0.658	0.520
128	0.694	0.512
512	0.711	0.524
2048	0.719	0.536
9192	0.722	

We see that when the singularity increases, the effectivity index is going down as predicted theoretically. (This is the influence of the term $\|u_0 - u_p\|_E$ in the estimate as discussed in Section 3.)

Let us now use an adaptively constructed mesh which is refined in the neighborhood of the origin where the singularity is located. Two sequences of meshes, A and B, which differ in the strengths of refinement were constructed.

The strength of the mesh refinement of the sequence B is stronger than that of A. In Fig. 4.3 we show two meshes of both sequences.

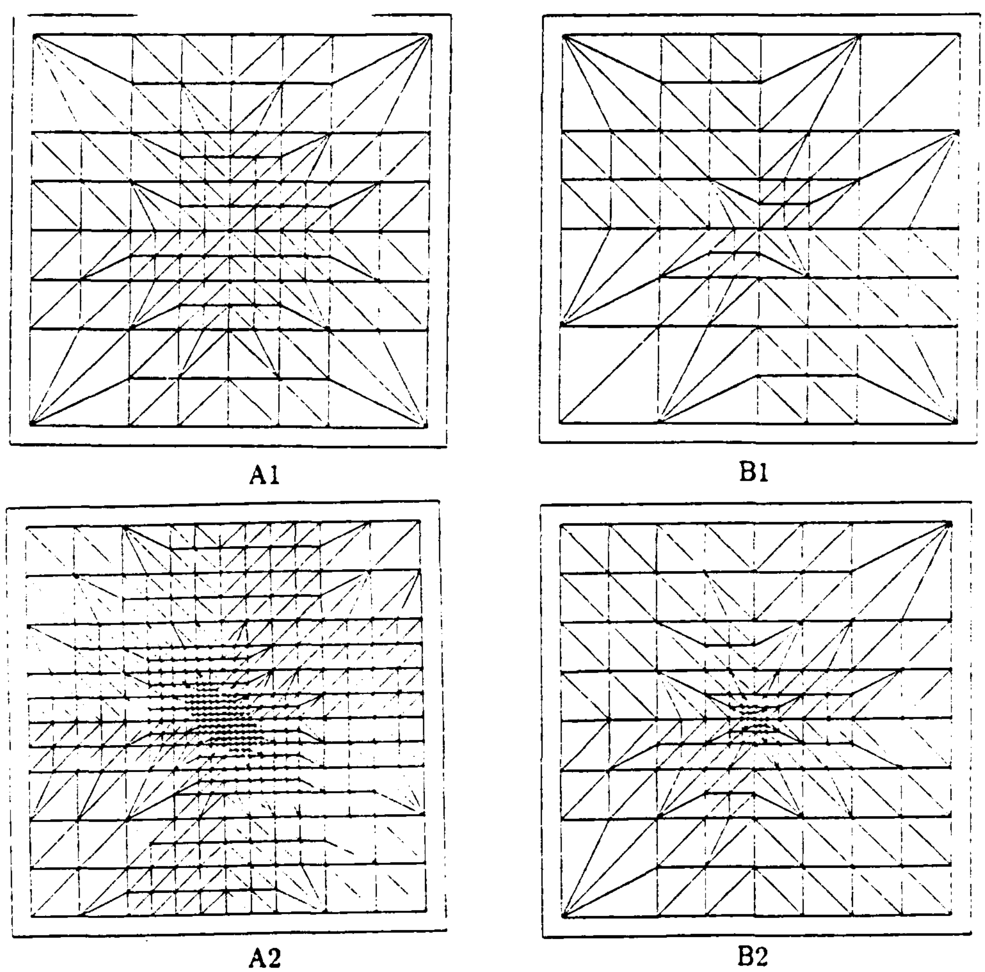

Fig. 4.3. The meshes of the sequences A and B.

Because of the stronger refinement in sequence B, a smaller value of p can be used and hence, the effectivity index will be better for sequence B than for A.

In Table 4.4 we report the effectivity index for Sequences A and B of the meshes for the case $\alpha = 0.5$

TABLE 4.4. The effectivity inces θ for Example 3 ($\alpha = 0.5$) for the meshes with different strength of refinement.

SEQUENCE A		SEQUENCE B	
NUMBER OF ELEMENTS	θ	NUMBER OF ELEMENTS	θ
32	0.658	32	0.658
87	0.712	64	0.719
194	0.738	96	0.787
370	0.761	168	0.830
725	0.787	244	0.880
1297	0.807	363	0.918
2346	0.828	485	0.960
		604	0.959
		768	0.975
		965	0.995

We see also that the effectivity index for mesh A is better than that for the uniform mesh (Table 4.3) as predicted by the theory.

We have seen that the theoretical results presented in Section 3 explain very well the computational data. For more numerical experiments, see [11].

Summarizing the results shown here and in [11], we can make the following conclusions about the estimator given in Section 3.

1. The estimator performs as expected from Theorem 3.1.

2. The effectivity index depends on the mesh, especially on the minimal angle of the triangles. Hence, avoiding small angles is preferable. The notion of the angle depends on the differential operator.

3. The effectivity index is not too sensitive to the topology of the mesh uniform or not uniform (except for the minimal angle).

4. The effectivity index can be larger or smaller than 1 depending on the solution.

5. If the solution has singular behavior, the quality of the estimator deteriorates and the effectivity index goes typically down. The derivation can be avoided by the proper refinement, for example, by adaptive procedure, as follows from Theorem 3.1.

5. ANALYSIS OF AN ESTIMATOR IN ONE DIMENSIONAL CASE

As we have said in Section 2, asymptotic exact estimators can be constructed by employing the superconvergence phenomena of the recovery of the gradient of the solution for linear elements or higher derivatives for elements of higher degree. This applies only for the meshes (essentially uniform) and solutions which produce the superconvergence effects. Nevertheless, these error estimators can be used then for wider class of meshes and solutions,and one can expect that a κ_1,κ_2, proper estimator will be obtained with acceptable values of κ_1,κ_2 for a reasonable class $(\mathcal{M},\mathcal{S})$. This is essentially the idea of [19].

We will address here a construction of an error estimator in one dimension which is directed to meshes constructed adaptively. For more details see [10].

Let us consider the model problem

$$(5.1) \qquad -u'' = f \quad \text{on} \quad I = (0,1)$$
$$u(0) = u(1) = 0$$

and let f be such that $u_0 \in H_0^1(I)$. Let $\mathcal{T}$ be a partition of the interval I with nodes $0 = x_0 < x_1 < \cdots < x_n = 1$, $\tau_i = [x_{i-i}, x_i]$, $h_i = x_i - x_{i-1}$, $i = 1,\ldots,n$. Consider the finite element method with piecewise linear elements on $\mathcal{T}$. In this case we have

$$(5.2) \qquad u_{s(\mathcal{T})}(x_i) = u_0(x_i).$$

Obviously $u'_{s(\tau)}$ is constant on τ_i and hence, it is discontinuous. Let us

now construct a continuous piecewise linear function U so that for $i = 1,\dots,n-1$,

$$(5.3)\qquad U(x_i) = u_s'|_{\tau_i} + \alpha_i \| u_s' \|_{\tau_i} = u_s'|_{\tau_{i+1}} + \beta_i \| u_s' \|_{\tau_i}$$

where

$$(5.4)\qquad \| u_s' \|_i = u_s'|_{\tau_{i+1}} - u_s'|_{\tau_i}$$

is the jump of the derivative of u_s at x_i (obviously $\alpha_i = 1 + \beta_i$), $U(x_i)$ depends only on the values of u_s' in the elements $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$, which contains the mesh point x_i. For the definition of $U(x_0)$ and $U(x_n)$, we will extend the $U(x)$ by an extension preserving the smoothness (for example, by an antisymmetric extension).

We will now define the error indicator

$$(5.5)\qquad \eta(\tau) = \left(\int_\tau (U(\alpha)-u_s')^2\, dx \right)^{1/2}$$

and accordingly the error estimator.

The problem is now how to select α_i. Let $\Delta_i = \dfrac{h_i}{h_{i+1}}$ and

$$(5.6)\qquad \alpha_{1,i} = \frac{1}{1+\Delta_i}$$

$$(5.7)\qquad \alpha_{2,i} = \frac{1}{1+\Delta_i^{-1}} .$$

The estimator $\mathcal{E}_1$ based on (5.6) was proposed in [28] and the estimator $\mathcal{E}_2$ based on (5.7) was proposed in [5] and [9]. Let us note that both estimators $\mathcal{E}_1$ and $\mathcal{E}_2$ are the same for uniform mesh. i.e., for $\Delta_i = 1$. We have

Theorem 5.1 (see [10])

i) $\mathcal{E}_2$ is an asymptotically exact estimator for any mesh and $u \in H^{2+\varepsilon}(I)$, $\varepsilon > 0$.

ii) $\mathcal{E}_1$ is an asymptotically exact estimator only for $\Delta = 1$, i.e., for the uniform mesh. □

Note that both estimators can be understood as extension of an asymptotically exact estimator for uniform mesh. Here the estimator $\mathcal{E}_2$ is asymptotically exact for arbitrary mesh, but similar result cannot be achieved in two dimensions.

Let us now address the problem of the optimal selection of α when u_0 is not sufficiently smooth and Theorem 5.1 is not applicable.

Let us assume that

i) $u''(x) > 0$ on I

ii) $u'(x) = u_s'|_{\tau_i} + a_i\left(x-x_{i-\frac{1}{2}}\right) + r_i(x)$ where $x_{i-\frac{1}{2}} = \dfrac{x_{i-1}+x_i}{2}$

and

$$a_i = -\frac{12}{h_i^3}\int_{\tau_i}(u-u_s)(x)dx$$

iii) $$(u'-u_s')|_{\tau_i}(x_{i-1}) = -\frac{a_ih_i}{2}(1-L_i), \qquad L_i = o(1)$$

$$(u'-u_s')|_{\tau_i}(x_i) = \frac{a_ih_i}{2}(1-R_i), \qquad R_i = o(1)$$

$$\int_{\tau_i}((u'-u_s')^2)dx = \frac{a_i^2h_i^3}{12}(1-K_i^2), \qquad K_i = o(1).$$

We remark that if u is sufficiently smooth L_i, R_i, $K_i = O(h_i)$. The assumption that $u''(x)$ does not change, the sign in the neighborhood of the singularity can be relaxed.

We have

Theorem 5.2 (see (10)). Assume that i), ii), and iii) are satisfied and that the mesh is equilibrated with respect to the energy norm (i.e., $\eta_i(\tau) \approx$ constant), then

(5.8) $$\alpha_i = \frac{1}{1+\Delta_0^{1/2}}$$

yield an asymptotically exact estimator.

Theorem 5.2 shows that the asymptotically exact estimator for an unsmooth solution and adaptively constructed meshes is different than the estimator which is asymptotically exact for smooth solution $\left(\alpha_i = \frac{1}{1+\Delta_0^{-1}}\right)$. When u_0 is smooth then, of course, $\Delta = 1 + o(1)$ for the adaptively constructed mesh.

Let us mention that the choice

(5.9) $$\alpha_i^{[s]} = \frac{1}{1+\Delta_i^{1/s}}$$

leads to the asymptotically optimal estimator when the mesh is equilibrated with respect to the norm $\|u'_s\|_{L_s}$, $1 \le s \le \infty$, and hence, the estimator $\mathcal{E}_1$ is related to the norm L_1 while the estimator $\mathcal{E}_2$ is not a good one for any adaptively constructed mesh for an unsmooth solution.

If we would know the values of $\left(\int_\tau (u-u_s)^2 dx\right)^{1/2} = \eta_{EX}(\tau)$, then the optimal error estimator will be (see [10]) for

(5.10) $$\hat{\alpha}_i^{[2]} = \frac{1}{1 + \frac{\eta_{EX}(\tau_{i+1})}{\eta_{EX}(\tau_i)}\,\Delta_i^{1/2}}$$

If

$$\frac{\eta_{EX}(\tau_{i+1})}{\eta_{EX}(\tau_i)} \approx \frac{\eta(\tau_{i+1})}{\eta(\tau_i)}$$

we can use these values in (5.10) and get the values of α_i which will be denoted by $\hat{\alpha}_i^{[2]}$.

Let us now present some examples.

Example 5.1. Let $u_0 = x^p - x$ be the solution of (5.1), and let the $x_i = \left(\frac{i}{n}\right)^{\beta}$, $i = 0,\dots,n$. Further, let $\Theta_i = \dfrac{\eta(\tau_i)}{\|u'-u'_s\|_{L_2(\tau_i)}}$ denote the elementary effectivity index. In Table 5.1 we report for $n = 10$ the value of $\|u' - u'_s\|_{L_2(\tau_i)}$ = error and the elementary effectivity indices Θ_i for the error estimator, based on $\alpha^{[1]}$, $\alpha^{[2]}$, $\hat{\alpha}^{[2]}$ and $\hat{\hat{\alpha}}^{[2]}$.

TABLE 5.1. The elementary effectivity index.

τ_i	$p = 0.75$ $\beta = 9.0$				
i	error	$\alpha^{[1]}$	$\alpha^{[2]}$	$\hat{\alpha}^{[2]}$	$\hat{\hat{\alpha}}^{[2]}$
1	1.99(-3)	1.290	1.237	1.095	1.131
2	8.11(-3)	1.118	0.881	1.011	0.895
3	1.42(-2)	1.128	0.861	0.907	0.909
4	1.95(-2)	1.110	0.897	0.929	0.934
5	2.42(-2)	1.084	0.928	0.950	0.957
6	3.86(-2)	1.063	0.948	0.964	0.962
7	3.27(-2)	1.048	0.961	0.973	0.981
8	3.66(-2)	1.037	0.940	0.979	0.971
9	4.04(-2)	1.029	0.976	0.984	0.998
10	4.40(-2)	0.700	0.936	0.986	0.968

τ_i	$p = 1.75$ $\beta = 2.5$				
i	error	$\alpha^{[1]}$	$\alpha^{[2]}$	$\hat{\alpha}^{[2]}$	$\hat{\hat{\alpha}}^{[2]}$
1	3.56(-4)	3.247	2.696	1.036	1.242
2	2.15(-3)	1.656	1.373	1.022	0.963
3	4.96(-3)	1.286	1.146	1.005	1.034
4	8.59(-3)	1.156	1.076	1.002	0.980
5	1.29(-2)	1.097	1.047	1.001	1.019
6	1.79(-2)	1.066	1.031	1.001	0.986
7	2.35(-2)	1.048	1.023	1.001	1.013
8	2.97(-2)	1.036	1.017	1.000	0.989
9	3.64(-2)	1.028	1.013	1.000	1.010
10	4.36(-2)	0.875	0.914	1.000	0.991

The meshes used are approximately equilibrated and hence the quality of the estimator based on $\alpha^{[1]}$ and $\alpha^{[2]}$ are comparable and of good quality. The optimally corrected estimator and the theoretical one (based on $\hat{\hat{\alpha}}^{[1]}$ and $\hat{\alpha}^{[2]}$ give a better effectivity index.

In Table 5.2 we show the elementary effectivity indices for the mesh which is very well equilibrated.

TABLE 5.2. The elementary effectivity index for an equilibrated mesh.

τ_i	$p = 1.75, \quad \beta = 6.5$		
i	error	$\alpha^{[1]}$	$\alpha^{[2]}$
1	1.50(-2)	1.05414	0.99399
2	1.57(-2)	0.95943	0.98301
3	1.57(-2)	0.99164	0.99574
4	1.57(-2)	0.99608	0.99809
5	1.57(-2)	0.99770	0.99890
6	1.57(-2)	0.99848	0.99928
7	1.57(-2)	0.99892	0.99949
8	1.57(-2)	0.99920	0.99962
9	1.57(-2)	0.99938	0.99971
10	1.57(-2)	0.99414	0.99974

We see from Table 5.2 that for perfectly equilibrated mesh, the estimator based on $\alpha^{[2]}$ performs very well, as expected.

Let us compare the performance of the estimator $\mathcal{E}_1$ and $\mathcal{E}_2$ based on (5.6 and (5.7)).

Example 5.2. Consider the mesh with $x_{i-2} = 0.01$, $x_{i-1} = 0.04$, $x_i = 0.09$, $x_{i+1} = 0.16$ and let $u(x) = \mathrm{Re}(x^{p+qi})$, and let us be interested in Θ_i for the estimators $\mathcal{E}_1$ and $\mathcal{E}_2$. Table 5.3 shows the result.

TABLE 5.3. The elementary error indicator Θ_i as function of p,q.

p	q = 0.0		q = 0.5	
	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_1$	$\mathcal{E}_2$
0.55	1.07	1.78	4.70	6.19
0.65	1.03	1.68	3.18	5.21
0.75	1.99	1.57	1.59	2.80
0.85	0.97	1.48	1.18	2.07
0.95	0.96	1.39	1.01	1.69
1.05	0.96	1.32	0.94	1.45
1.15	0.96	1.26	0.92	1.29
1.25	0.97	1.20	0.93	1.17
1.35	0.98	1.15	0.97	1.08
1.45	1.01	1.11	1.01	1.02
1.55	1.03	1.07	1.07	0.99
1.65	1.07	1.04	1.13	0.97
1.75	1.10	1.02	1.20	0.96
1.85	1.15	1.01	1.27	0.97
1.95	1.19	1.00	1.35	0.99
2.05	1.24	1.00	1.44	1.02
2.15	1.30	1.01	1.52	1.06
2.25	1.35	1.02	1.61	1.11

We see from this table that the estimator $\mathcal{E}_1$ is better than $\mathcal{E}_2$ for an unsmooth solution, and $\mathcal{E}_2$ is better than $\mathcal{E}_1$ for a smooth solution, as expected, due to the asymptotic exactness of $\mathcal{E}_2$.

We can make the following conclusion:

1) The asymptotically exact estimator for the set of smooth solution does not perform well for an unsmooth solution.

2) The asymptotically exact estimator for the class $\mathcal{M}^*$ of equilibrated meshes performs well also for an unsmooth solution. This estimator is a different one when compared with the asymptotically exact estimator for smooth solution.

3) For different equilibration criteria different estimators are asymptotically optimal.

Although no two dimensional analysis is available, we expect that very good estimators are possible to construct for equilibrated meshes as in one dimension.

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